Journal of Engineering Physics and Thermophysics, Vol. 75, No. 1, 2002

METHOD OF CHARACTERISTICS FOR ANISOTROPIC BODIES WITH A FINITE VELOCITY OF PROPAGATION OF HEAT

M. D. Martynenko[a] **and S. M. Bosyakov**[b]

UDC 539.3

The equation of propagation of the characteristic surfaces in anisotropic continuous media with a finite time of relaxation of the heat flux is obtained.

The regularities of existence of waves in an anisotropic thermoelastic medium with a finite velocity of propagation of heat have been considered quite adequately in [1–3] in the context of plane singular waves. Below we analyze the regularities of propagation of heat waves in anisotropic media with relaxation of the heat flux based on the general method of characteristics. This enables us to determine the kinematic characteristics of the wave and to find the equation of the wave surface, in particular, to calculate the group velocity of propagation of the wave which is the velocity of propagation of the temperature disturbances. Based on the formulas obtained, we have calculated the velocities of propagation of the heat waves in certain anisotropic bodies as functions of the angle of inclination of the normal to the wave surface. This is especially important, since most structural materials are anisotropic.

Propagation of Removable-Discontinuity Surfaces. To describe wave processes initiated by the presence of thermal fields we use the hyperbolic law of heat conduction in the form [4, 5]

$$\sum_{i,j=1}^{3} \lambda_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} - c_v \left(\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} \right) = 0 \,, \tag{1}$$

where λ_{ij} is the thermal-conductivity coefficient (number of its components is equal to the number of components of the tensor of thermoelastic stresses [6]), T is the absolute temperature, c_v is the heat capacity at constant volume, and τ is the relaxation time of the heat flux, $i, j = \overline{1, 3}$. Let us consider the propagation of removable discontinuities (acceleration waves) that occur in the case where the second derivatives of the temperature T experience discontinuity in passage through the surface

$$Z\,(t, X) = \text{const} \,. \tag{2}$$

In the general case, on the same surface we specify the initial data for solution of the Cauchy problem and pass to new variables according to the formulas

$$Z = Z\,(t, X) \,, \quad Z_k = Z_k\,(t, X) \,, \quad k = \overline{1, 3} \,. \tag{3}$$

The derivatives with respect to the previous variables have the following form:

[a]Belarusian State University, Minsk, Belarus; [b]Brest State Technical University, Brest, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 1, pp. 37–40, January–February, 2002. Original article submitted April 12, 2000; revision submitted June 21, 2001.

$$\frac{\partial T(t, X)}{\partial x_k} = \sum_{i=0}^{3} \frac{\partial T}{\partial Z_i} \frac{\partial Z_i}{\partial x_k},$$

$$\frac{\partial^2 T}{\partial x_k \partial x_n} = \sum_{i,j=0}^{3} \frac{\partial^2 T}{\partial Z_j \partial Z_i} \frac{\partial Z_i}{\partial x_k} \frac{\partial Z_j}{\partial x_n} + \sum_{i=0}^{3} \frac{\partial T}{\partial Z_i} \frac{\partial^2 Z_i}{\partial x_n \partial x_k},$$

$$Z \equiv Z_0, \quad t \equiv x_0. \tag{4}$$

We introduce (4) into Eq. (1) and write only those terms that contain the derivatives $\partial^2 T/\partial Z^2$ [7, 8]. We obtain

$$\left(\sum_{i,j=1}^{3} \lambda_{ij} p_i p_j - c_v \tau p_0^2 \right) \frac{\partial^2 T}{\partial Z^2} + \ldots = 0, \tag{5}$$

where $p_0 = \partial Z/\partial t$ and $p_k = \partial Z/\partial x_k$, $k = \overline{1, 3}$.

The equation of the characteristic surface will be found from the condition of unsolvability of the last equation relative to the derivative $\partial^2 T/\partial Z^2$ [7, 8]. We will have

$$\sum_{i,j=1}^{3} \lambda_{ij} p_i p_j - c_v \tau p_0^2 = 0. \tag{6}$$

We divide (6) by $g^2 = p_1^2 + p_2^2 + p_3^2$. Taking into account that the velocity of propagation of the wave surface and the direction cosines of the normal to the characteristic surface are determined, respectively, by the formulas $V = p_0/g$ and $\cos \alpha_i = p_i/g$, $i = \overline{1, 3}$ [7, 8], we obtain

$$\sum_{i,j=1}^{3} \lambda_{ij} \cos \alpha_i \cos \alpha_j - c_v \tau V^2 = 0. \tag{7}$$

As is indicated in [6], the number of constants λ_{ij} in the most general case of thermal anisotropy (triclinic system of symmetry) is equal to three: $\lambda_{11} = \lambda_1$, $\lambda_{22} = \lambda_2$, and $\lambda_{33} = \lambda_3$. Therefore, Eq. (6) can be written in the form

$$V^2 = \beta^2 \left(\lambda_1 \cos^2 \alpha_1 + \lambda_2 \cos^2 \alpha_2 + \lambda_3 \cos^2 \alpha_3 \right), \quad \beta^2 = \frac{1}{c_v \tau}.$$

By specifying the direction of propagation of the discontinuity surface (values of $\cos \alpha_i$, $i = \overline{1, 3}$) we can easily calculate the velocity of its propagation for different anisotropic bodies. Thus, setting the relaxation time of the heat flux for metals and nonmetals $\tau \sim 10^{-11}$ sec and $\tau \sim 10^{-13}$ sec [2, 5], we calculate the velocities of propagation of the heat wave for crystals of the trigonal and hexagonal systems of symmetry in the plane ($\sin \alpha$, 0, $\cos \alpha$) (Table 1) based on the last formula [9, 10].

The thermal conductivity of cubically anisotropic bodies is the same in all directions [6]; therefore, formula (7) will be written in the form

$$V = \sqrt{\frac{\lambda}{c_v \tau}}, \quad \lambda_{11} = \lambda_{22} = \lambda_{33} = \lambda.$$

TABLE 1. Velocities of Propagation of the Heat Waves in Crystals of the Hexagonal and Trigonal Systems of Symmetry as Functions of the Angle of Inclination of the Wave Normal in the Plane (sin α, 0, cos α)

Material	T, °C	Thermal conductivity, J/(m·sec·deg)		$c_V \cdot 10^3$, J/(m^3·deg)	Velocity V, m/sec, at different angles of inclination of the wave normal, deg						
		$\lambda_1 = \lambda_2$	λ_3		0	15	30	45	60	75	90
Quartz	30	6.5	11.3	1680.7	8200	8082	7752	7275	6769	6371	6219
Calcite	30	4.2	5.0	2434.1	4532	4508	4441	4347	4252	4180	4154
Bismuth	30	9.2	6.6	1190.3	745	754	780	815	848	871	879
Graphite	14	355	89	1599.4	2360	2584	3120	3726	4247	4591	4711

TABLE 2. Velocities of Propagation of the Heat Waves in Cubically Anisotropic Bodies

Material	T, °C	λ, J/(m·sec·deg)	$c_V \cdot 10^3$, J/(m^3·deg)	Velocity V, m/sec
Aluminum	30	208	2435	2923
Copper	30	410	3404.8	3470
Magnesium	25	170	1810	3065
Gold	100	310	2512	3513
Silver	300	362	2631	3710
Germanium	–100	105	1407	2732
Lithium	182	71	2163	1812
Molybdenum	327	158	2470	2530
Nickel	100	83	4279	1400
Lead	300	28,14	1551	1370
Tungsten	327	126	2645.3	2183
Silicon	0	167	1572.6	3260

The values of the velocities of propagation of the heat wave are given in Table 2.

As is seen from Tables 1 and 2, the velocities of propagation of the heat waves depend on $\sqrt{\tau}$ in inverse proportion; as τ decreases, the velocity increases significantly. In the classical case ($\tau \to 0$), this velocity increases indefinitely, which is in good agreement with [1, 2].

Propagation of Nonremovable Discontinuities. The method of characteristics makes it possible to investigate the regularities of propagation of nonremovable-discontinuity surfaces (velocity waves) when the partial derivatives of first order of T experience discontinuity (discontinuity of the first kind) on $Z(t, X)$ = const, while the function itself remains continuous.

We denote the limiting values of the first derivatives at an arbitrarily selected point $N(x_1, x_2, x_3, t)$ of the surface $Z(t, X)$ = const by $\partial T^+/\partial x_k$, $\partial T^+/\partial t$, $\partial T^-/\partial x_k$, and $\partial T^-/\partial t$. We fix this point and consider the point $(N + dN)$ $(x_1 + \partial x_1,\ x_2 + dx_2,\ x_3 + \partial x_3,\ t)$ which is close to it. Accurate to infinitesimals of higher order we will have [8]

$$\sum_{k=1}^{3} \frac{\partial Z}{\partial x_k} dx_k + \frac{\partial Z}{\partial t} dt = 0 , \tag{8}$$

where $\partial Z/\partial x_k$ and $\partial Z/\partial t$, $k = \overline{1, 3}$, are calculated at point N. Taking into account that these derivatives are continuous on both sides of the surface (2) up to the points of the surface itself and the continuity of T, we obtain [8]

$$T(N+dN)-T(N)=\sum_{k=1}^{3}\frac{\partial T^{+}}{\partial x_k}dx_k+\frac{\partial T^{+}}{\partial t}dt+O_1(dx_k,\ dt)=\sum_{k=1}^{3}\frac{\partial T^{-}}{\partial x_k}dx_k+\frac{\partial T^{-}}{\partial t}dt+O_2(dx_k,\ dt)\ .$$

It follows accurate to infinitesimals of higher order that

$$\sum_{k=1}^{3}\left(\frac{\partial T^{+}}{\partial x_k}-\frac{\partial T^{-}}{\partial x_k}\right)dx_k+\left(\frac{\partial T^{+}}{\partial t}-\frac{\partial T^{-}}{\partial t}\right)dt=0\ . \tag{9}$$

As is indicated in [8], relations (9) must be fulfilled for arbitrary ∂x_k and ∂t which satisfy (8). Therefore, from (9) we have

$$p_k\frac{\partial T^{+}}{\partial t}-p_0\frac{\partial T^{+}}{\partial x_k}=p_k\frac{\partial T^{-}}{\partial t}-p_0\frac{\partial T^{-}}{\partial x_k}\ ,\quad k=\overline{1,3}\ . \tag{10}$$

It follows from (10) that [8]

$$\frac{\partial T}{\partial t}p_k-\frac{\partial T}{\partial x_k}p_0=M_k\ , \tag{11}$$

where M_k, $k=\overline{1,3}$, is a continuous function.

Dynamic compatibility conditions that will be obtained from the laws of conservation must be fulfilled in addition to the kinematic conditions. Following [8], we write the equality which is a consequence of the condition of heat balance [5]:

$$c_vV\left(\left(1+\tau\frac{\partial T}{\partial t}\right)^{-}-\left(1+\tau\frac{\partial T}{\partial t}\right)^{+}\right)=\sum_{j=1}^{3}\lambda_{kj}\frac{\partial T^{-}}{\partial x_j}\cos\alpha_j-\sum_{j=1}^{3}\lambda_{kj}\frac{\partial T^{+}}{\partial x_j}\cos\alpha_j\ . \tag{12}$$

If we group the terms relating to different sides of the surface (2), then from (12) we obtain

$$\sum_{j=1}^{3}\lambda_{kj}\frac{\partial T^{+}}{\partial x_j}\cos\alpha_j-c_vV\left(1+\tau\frac{\partial T}{\partial t}\right)^{+}=\sum_{j=1}^{3}\lambda_{kj}\frac{\partial T^{-}}{\partial x_j}\cos\alpha_j-c_vV\left(1+\tau\frac{\partial T}{\partial t}\right)^{-}\ . \tag{13}$$

In order to describe the dynamic compatibility conditions (13) in final form it should be taken into account that $V=p_0/g$ and $\cos\alpha_j=p_j/g$, $j=\overline{1,3}$. Then

$$\sum_{j=1}^{3}\lambda_{jk}\frac{\partial T}{\partial x_j}p_j-c_vp_0\left(1+\tau\frac{\partial T}{\partial t}\right)=M_4\ , \tag{14}$$

where M_4 is a continuous function.

System (11) and (14) makes it possible to determine all the derivatives of first order of the function T. In order to simplify the calculation we reduce it to a simpler form. To do this we multiply Eq. (8) by p_0 and replace the resultant expressions $p_0\frac{\partial T}{\partial x_k}$, $k=1, 3$, by the left-hand sides of the equalities

$$\frac{\partial T}{\partial t} p_k - M_k = \frac{\partial T}{\partial x_k} p_0 , \quad k = \overline{1, 3} .$$

As a result we will have

$$\sum_{k,j=1}^{3} \frac{\partial T}{\partial t} \lambda_{kj} p_k p_j - c_v \left(p_0^2 + \tau p_0^2 \frac{\partial T}{\partial t} \right) + \sum_{k,j=1}^{3} \lambda_{kj} M_j p_j p_0 = M_4 p_0 . \tag{15}$$

The unsolvability of Eq. (15) relative to $\partial T/\partial t$ yields the necessary condition that the partial derivative of first order $\partial T/\partial t$ has a discontinuity of the first kind on the surface (2). Hence, to find the equation of the nonremovable-discontinuity surface we equate the coefficient of this derivative in (15) to zero. As a result, we arrive at Eq. (6). Thus, the field of the temperature T with nonremovable discontinuities of partial derivatives of the first kind at points of the surface $Z(t, X)$ = const exists in the case where this surface turns out to be characteristic for Eq. (1). Generally speaking, the opposite is incorrect, i.e., it cannot be stated that the nonremovable-discontinuity surface is characteristic for the given equation. This conclusion is in agreement with the analysis of [7, 8].

Bicharacteristics. In order to obtain the equation of the characteristic surface, we express p_0 from (6):

$$p_0 = \beta \sqrt{\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2} , \quad \beta^2 = \frac{1}{c_v \tau} .$$

This yields the following equations for bicharacteristics [7, 8]:

$$\frac{dx_k}{dt} = \frac{\beta \lambda_k p_k}{\sqrt{\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2}} , \quad k = \overline{1, 3} ,$$

where the right-hand side involves three parameters, p_1, p_2, and p_3, which are independent of the time t. Then, setting t = 1, we obtain

$$x_k = \frac{\beta \lambda_k p_k}{\sqrt{\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2}} , \quad k = \overline{1, 3} . \tag{16}$$

From (16), upon obvious transformations, we will have the following equation of the characteristic (wave) surface:

$$\frac{x_1^2}{\lambda_1} + \frac{x_2^2}{\lambda_2} + \frac{x_3^2}{\lambda_3} = \beta^2 .$$

In the case of cubic anisotropy ($\lambda_1 = \lambda_2 = \lambda_3$), the last equation becomes the equation of a sphere.

In conclusion, we note that despite the existing theoretical foundation of the method of characteristics, it has not yet been applied to investigation of the transient processes in generalized heat conduction of anisotropic bodies.

REFERENCES

1. A. G. Shashkov, V. A. Bubnov, and S. Yu. Yanovskii, *Wave Phenomena of Heat Conduction: System Structural Approach* [in Russian], Minsk (1993).

2. J. Sharma and H. J. Singh, *J. Acoust. Soc. Am.*, **85**, No. 4, 1407–1413 (1989).
3. J. Ignaczak, *J. Therm. Stresses*, **8**, No. 1, 25–40 (1985).
4. J. B. Haddow and J. L. Wegner, *Math. Mech. Solids*, **1**, No. 1, 111–127 (1996).
5. Yu. M. Kolyano, B. V. Koval'chuk, and I. O. Goi, *Dinam. Prochn. Mashin*, No. 52, 117–127 (1991).
6. L. D. Landau and E. M. Lifshits, *Elasticity Theory* [in Russian], Moscow (1986).
7. V. I. Smirnov, *A Course in Higher Mathematics* [in Russian], Vol. 4, Pt. 2, Moscow (1981).
8. G. I. Petrashen', in: *Problems of the Dynamic Theory of Propagation of Seismic Waves* [in Russian], Issue 18, Leningrad (1978), pp. 5–278.
9. *Modern Crystallography*, Vol. 4, *Physical Properties of Crystals* [in Russian], Moscow (1981).
10. I. M. Kikoin (ed.), *Tables of Physical Quantities. Handbook* [in Russian], Moscow (1976).